A NOTE ON ONE OF THE METHODS OF SOLUTION OF THE DIFFUSION EQUATION

I. Ya. Kolesnik

Inzhenerno-Fizicheskii Zhurnal, Vol. 12, No. 2, pp. 234–239, 1967

UDC 536.421.3+532.72

An account is given of one of the methods of solution of the equation which describes the variation of the radius of a water droplet whose temperature has become established.

In the very first time interval following introduction of water droplets into an air medium the temperature and concentration fields become established. During this time, as may easily be verified, the droplet radius may be considered constant.

Following establishment of the temperature and diffusion fields the droplets evaporate or grow at the psychrometric temperature, and at this stage it is undoubtedly necessary to take account of the variation of radius with time.

This kind of subdivision of the time interval has a physical basis and is very often applied in the solution of problems of this kind [1, 2].

The present paper is devoted to solving the equation for mass transfer through the drop surface, which equation may be written in the form

$$\frac{dm}{dt} = DS \frac{\partial \rho_c}{\partial r}\bigg|_{r=R}$$

or

$$\gamma \frac{dR}{dt} = D \frac{\partial \rho_c}{\partial r}\bigg|_{r=R}. \tag{1}$$

The initial condition for (1) is

$$R = R_0 \quad \text{at} \quad t = 0.$$

Since ρ_c, generally speaking, is a function of t and r, we must add to (1) a further equation describing the density variation for water vapor in air. Bearing in mind that the problem under consideration is spherically symmetrical, this equation has the form

$$\frac{\partial \rho_c}{\partial t} = D\left(\frac{\partial^2 \rho_c}{\partial r^2} + \frac{2}{r}\,\frac{\partial \rho_c}{\partial r}\right). \tag{2}$$

The boundary conditions for (2) are

$$\rho_c = \rho_s \quad \text{at } r = R,$$

$$\rho_c = \rho_c^0 \quad \text{at} \quad t = 0. \tag{3}$$

Going over to the function u in (2) and (3) via the formula

$$u = r\rho_c,$$

we obtain

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial r^2}, \tag{4}$$

$$u = \rho_s R \quad \text{at} \quad r = R,$$

$$u = r\rho_c^0 \quad \text{at} \quad t = 0. \tag{5}$$

To solve the problem given by (4) and (5) we shall examine the equation

$$\frac{d^2V}{dr^2} = \alpha^2 V.$$

It is clear that

$$V = \exp(-\alpha r) \tag{6}$$

is one special solution of the equation.

Multiplying the left and right sides of (4) by (6), and integrating with respect to r over the range R to ∞, we obtain after simple transformations

$$\frac{\partial}{\partial t}\int_R^\infty u\exp(-\alpha r)\,dr - u|_{r=R}\exp(-\alpha R)\,\frac{dR}{dt} = -D\left(\frac{\partial u}{\partial r} + \alpha u\right)\bigg|_{r=R}\exp(-\alpha R) + D\alpha^2\int_R^\infty u\exp(-\alpha r)\,dr. \tag{7}$$

Introducing the notation

$$\bar{u} = \int_R^\infty u\exp(-\alpha r)\,dr$$

and supposing that $\operatorname{Re}\alpha > 0$, we bring (7) to the form

$$\exp(D\alpha^2 t)\,\frac{\partial}{\partial t}\exp(-D\alpha^2 t)\,\bar{u} = -\left(D\frac{\partial u}{\partial r} + D\alpha u - u\frac{dR}{dt}\right)\bigg|_{r=R}\exp(-\alpha R). \tag{8}$$

After multiplying (8) by $\exp(-D\alpha^2 t)$ and then integrating it from 0 to t, we have

$$\bar{u}\exp(-D\alpha^2 t) = -\int_0^t\left(D\frac{\partial u}{\partial r} + D\alpha u - u\frac{dR}{dt}\right)\bigg|_{r=R}\exp(-\alpha R)\exp(-D\alpha^2 t)\,dt + \bar{u}|_{t=0}.$$

Going to the limit as $t \to \infty$ in this last equation and using the boundary condition (5) and the fact that

$$\bar{u}|_{t=0} = \frac{\rho_c^0}{\alpha^2}(1 + R_0\alpha)\exp(-\alpha R_0),$$

$$\frac{\partial u}{\partial r}\bigg|_{r=R} = \rho_s + \frac{\gamma}{D}R\frac{dR}{dt},$$

we obtain

$$\int_0^\infty\left(D\rho_s + \gamma R\frac{dR}{dt} + D\alpha R\rho_s - R\rho_s\frac{dR}{dt}\right)\times \tag{9}$$

$$\times \exp(-\alpha R)\exp(-D\alpha^2 t)\,dt =$$

$$= \frac{\rho_c^0}{\alpha^2}(1+\alpha R_0)\exp(-\alpha R_0). \qquad (9)\ \text{(cont'd)}$$

If we examine relatively large drops, it may be considered that the density of saturated vapors at the drop surface does not depend on the drop radius and is approximately equal to the density of saturated vapor above a plane water surface.

Bringing into consideration a dimensionless parameter Ko, defined by the formula

$$\mathrm{Ko} = \rho_s/\gamma,$$

and taking into account that the quantities ρ_c^0 and ρ_s are commensurate, which means

$$\rho_c^0/\gamma = \mu\,\mathrm{Ko}, \quad (\mu = \rho_c^0/\rho_s),$$

after division by γ we may write (9) in the form

$$\int_0^\infty \left\{D\,\mathrm{Ko} + R\frac{dR}{dt} + \sqrt{D}\sqrt{s}\,R\,\mathrm{Ko} - \mathrm{Ko}\,R\frac{dR}{dt}\right\} \times$$

$$\times \exp\left(-\frac{\sqrt{s}}{\sqrt{D}}R\right)\exp(-st)\,dt =$$

$$= \mathrm{Ko}\,\mu\left(\frac{\sqrt{D}R_0}{\sqrt{s}} + \frac{D}{s}\right)\exp\left(-\frac{\sqrt{s}}{\sqrt{D}}R_0\right), \qquad (10)$$

where

$$s = D\alpha^2.$$

We note that the parameter Ko is of the order 10^{-3}–10^{-5}.

It is expedient to seek a solution of (10) in the form of a series with respect to Ko:

$$R = r_0 + \mathrm{Ko}\,r_1 + \mathrm{Ko}^2 r_2 + \ldots\ . \qquad (11)$$

Giving attention to the initial condition for (1), we shall consider that the following equalities obtain:

$$r_0|_{t=0} = R_0, \quad r_n|_{t=0} = 0 \quad (n = 1, 2, 3, \ldots). \qquad (12)$$

Putting Ko = 0 in (10) and (11), we obtain for r_0 the equation

$$\int_0^\infty r_0\frac{dr_0}{dt}\exp\left(-\frac{\sqrt{s}}{\sqrt{D}}r_0\right)\exp(-st)\,dt = 0.$$

From the fact that the transform is zero it follows that the original in the class of continuous functions must also equal zero. Hence, using (12), we obtain

$$r_0 = R_0. \qquad (13)$$

After expansion of the exponent in a series, taking (11) into account, and after equating terms with Ko at the first power in Eq. (10) we shall have the following equation for r_1:

$$\int_0^\infty (D + R_0\dot{r}_1 +$$

$$+ \sqrt{Ds}R_0)\exp(-st)\,dt = \mu\left(\frac{R_0\sqrt{D}}{\sqrt{s}} + \frac{D}{s}\right),$$

from which it is easy to obtain, following integration,

$$\bar{r}_1 = \sqrt{D}(\mu-1)\frac{1}{s\sqrt{s}} + \frac{D(\mu-1)}{R_0}\frac{1}{s^2}.$$

In the space of the originals, r_1 has the form

$$r_1 = 2\sqrt{D}(\mu-1)\sqrt{\frac{t}{\pi}} + \frac{D(\mu-1)}{R_0}t. \qquad (14)$$

Equating coefficients of Ko^2 in (10) and transferring to the originals' space, we obtain, after simple transformations,

$$r_2 = 2r_1 - \frac{r_1^2}{2R_0} + \frac{1}{2\sqrt{D}}\frac{1}{\sqrt{\pi t}} * \frac{dr_1^2}{dt}. \qquad (15)$$

It follows from (14) that $r_1\dot{r}_1$ may be written in the form

$$r_1\dot{r}_1 = \frac{2D(\mu-1)^2}{\pi} + \frac{3D\sqrt{D}(\mu-1)^2}{\sqrt{\pi}R_0} \times$$

$$\times \sqrt{t} + \frac{D^2(\mu-1)^2}{R_0^2}t. \qquad (16)$$

Taking (16) into account, the convolution on the right side of (15) may be represented as follows:

$$\frac{1}{2\sqrt{D}}\frac{1}{\sqrt{\pi t}} * \frac{dr_1^2}{dt} =$$

$$= \frac{2\sqrt{D}(\mu-1)^2}{\pi}\frac{1}{\sqrt{\pi t}} * 1 +$$

$$+ \frac{3D(\mu-1)^2}{\sqrt{\pi}R_0}\frac{1}{\sqrt{\pi t}} * \sqrt{t} +$$

$$+ \frac{D\sqrt{D}(\mu-1)^2}{R_0^2}\frac{1}{\sqrt{\pi t}} * t. \qquad (17)$$

Since

$$\frac{1}{\sqrt{\pi t}} * 1 \to \frac{1}{s\sqrt{s}},$$

$$\frac{1}{\sqrt{\pi t}} * \sqrt{t} \to \frac{\sqrt{\pi}}{2}\frac{1}{s^2},$$

$$\frac{1}{\sqrt{\pi t}} * t \to \frac{1}{s^2\sqrt{s}},$$

then it is true that

$$\frac{1}{\sqrt{\pi t}} * 1 = \frac{2}{\sqrt{\pi}}\sqrt{t}, \quad \frac{1}{\sqrt{\pi t}} * \sqrt{t} = \frac{\sqrt{\pi}}{2}t,$$

$$\frac{1}{\sqrt{\pi t}} * t = \frac{4}{3\sqrt{\pi}}t\sqrt{t}. \qquad (18)$$

Using (14), (17), and (18), we shall write (15) in the form

$$r_2 = \frac{4\sqrt{D}(\mu-1)}{\sqrt{\pi}}\left(1 + \frac{\mu-1}{\pi}\right) \times$$

$$\times \sqrt{t} + \frac{2D(\mu-1)}{R_0}\left[1 - \frac{\mu-1}{\pi} + \frac{3}{4}(\mu-1)\right]t -$$

$$- \frac{2}{3}\frac{D\sqrt{D}(\mu-1)^2}{\sqrt{\pi}R_0^2}t\sqrt{t} - \frac{D^2(\mu-1)^2}{2R_0^3}t^2.$$

We note that r_3, r_4, ... may be found from (10) by equating coefficients, respectively, of Ko^3, Ko^4, ..., and by subsequently solving the equations obtained.

By replacing r_1, r_2, ... in (11) by expressions (14), (15), ..., respectively, we obtain the solution of the problem being examined in the form of a power series in the dimensionless parameter Ko.

In the dimensionless variables, introduced by the formulas

$$r_n = R_0 \nu_n \ (n = 0, 1, 2, \ldots), \quad t = t_0 \tau,$$

where $t_0 = R_0^2/D$ is a characteristic time, we have

$$\nu_0 = 1,$$

$$\nu_1 = (\mu - 1)\left(\tau + \frac{2}{\sqrt{\pi}}\sqrt{\tau}\right),$$

$$\nu_2 = (\mu - 1)\left\{\frac{4}{\sqrt{\pi}}\left(1 - \frac{\mu - 1}{\sqrt{\pi}}\right)\sqrt{\tau} + \right.$$

$$+ 2\left[1 - \frac{\mu - 1}{\pi} + \frac{3}{4}(\mu - 1)\right]\tau -$$

$$\left. - \frac{2}{3\sqrt{\pi}}(\mu - 1)\tau\sqrt{\tau} - \frac{1}{2}(\mu - 1)\tau^2\right\},$$

$$\nu = 1 + Ko\,\nu_1 + Ko^2\nu_2 + Ko^3\nu_3 + \ldots.$$

For large τ

$$\nu \sim 1 + Ko(\mu - 1)\tau - \frac{1}{2}Ko^2(\mu - 1)^2\tau^2 + \ldots.$$

Thus, this method of expanding the solution in terms of a small parameter and using a Laplace transformation permits us to solve the equation giving the variation of water drop radius with time for established temperature.

NOTATION

m is the mass of a water drop; t is the time; D is the diffusion coefficient for water vapor in air; S is the surface area of a drop; ρ_c is the density of water vapor in the air; R is the drop radius; r is the spatial coordinate; ρ_s is the density of saturated vapor at the droplet temperature; γ is the density of the drop; Ko is the dimensionless parameter; μ is the supersaturation of the air by the water vapor; s is the Laplace transformation variable; r_n $(n = 0, 1, 2, \ldots)$ are the coefficients in the expansion of the drop radius in terms of powers of Ko; ν is the dimensionless drop radius; τ is the dimensionless time; t_0 is the characteristic time.

REFERENCES

1. M. V. Buikov and S. S. Dukhin, IFZh, 5, no. 3, 1962.
2. M. V. Buikov, IFZh, 5, no. 4, 1962.

31 January 1966

Institute for Ventilation, Dust Precipitation, and Air Purification for the Metallurgy Industry, Krivoi Rog